

Complete Surfaces with Ends of Non Positive Curvature

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May 6, 2014

Abstract

In this paper we extend Efimov's Theorem by proving that any complete surface in \mathbb{R}^3 with Gauss curvature bounded above by a negative constant outside a compact set has finite total curvature, finite area and is properly immersed. Moreover, its ends must be asymptotic to half-lines. We also give a partial solution to Milnor's conjecture by studying isometric immersions in a space form of complete surfaces which satisfy that outside a compact set they have non positive Gauss curvature and the square of a principal curvature function is bounded from below by a positive constant.

1 Introduction

An important part in the study of complete surfaces of non positive curvature in \mathbb{R}^3 has been directed at nonexistence of isometric immersions. The investigation of the isometric immersion of metrics with negative curvature goes back to Hilbert. He proved in 1901, see [12, 13], that the full hyperbolic plane can not be isometrically immersed in \mathbb{R}^3 . This means, it is impossible to extend a regular piece of a surface of constant negative curvature without the appearance of singularities. Hilbert's theorem has attracted the attention of many mathematician and an important amount of work has been done in this direction (see for example [3, 4, 5, 9, 11, 14, 22, 27, 28]).

A next natural step, conjectured by many geometers [8] was to extend such a result to complete surfaces which Gauss curvature is bounded above by a negative constant. The final solution to this problem was obtained by Efimov in 1963 more than sixty years later, see [9, 16]. Efimov proved that no surface can be \mathcal{C}^2 -immersed in the Euclidean 3-space so as to be complete in the induced Riemannian metric, with Gauss curvature $K \leq \text{const} < 0$. In the following years Efimov extended this result in several ways, see [10].

Although Efimov's proof is very delicate, it is ingenious and does not depend upon sophisticated or modern techniques. In fact, it is derived from a general result about \mathcal{C}^1 -immersions between planar domains, see [9, Lemma 1] and [16, Main Lemma]. Here, we will show that the same method can be applied to study complete surfaces which Gauss curvature is negative and bounded away from zero outside a compact subset. We shall prove, see Theorem 5, that:

Research partially supported by Ministerio de Educación Grant No. MTM2010-19821, Junta de Andalucía Grants No. FQM325, No. P06-FQM-01642.

Any complete \mathcal{C}^2 -immersed surface in \mathbb{R}^3 with Gauss curvature satisfying $K \leq \text{const} < 0$ outside a compact subset, has finite total curvature, finite area and is properly immersed. Moreover, its ends are asymptotic to half-lines, see Figure 1.

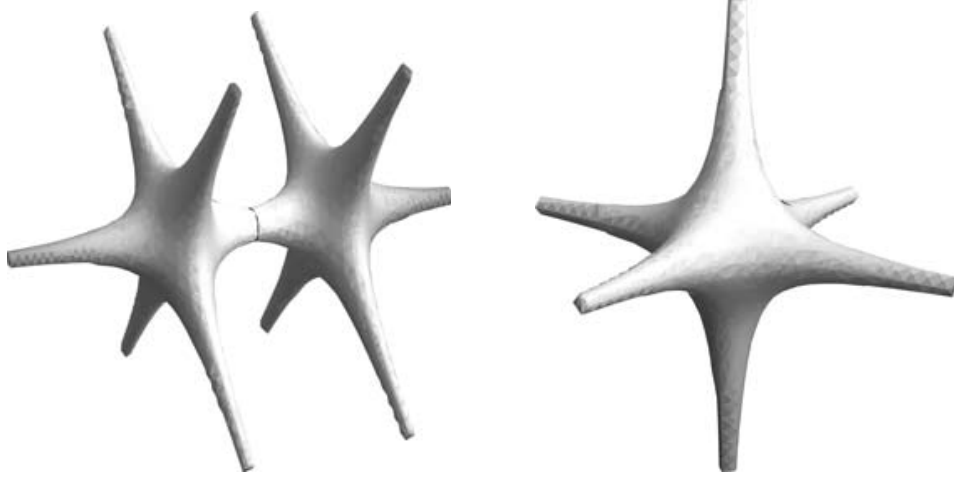


Figure 1: Complete surfaces with negatively curved ends.

A more general result than the above mentioned is indeed proven, see Theorem 3 and Theorem 4.

Despite the considerable progress in the understanding of negatively curved surfaces, important questions suggested by Hilbert's Theorem remain unanswered to this day. Among the most interesting open problems we mention the following conjecture due to John Milnor, (see [20]),

MILNOR'S CONJECTURE. *Assume $\psi : \Sigma \longrightarrow \mathbb{R}^3$ is a complete, umbilic free immersion so that their principal curvatures k_1 and k_2 satisfy*

$$k_1^2 + k_2^2 \geq \text{const} > 0.$$

Then either the Gauss curvature K of ψ changes sign or else $K \equiv 0$.

In this paper, we also aim at taking a small step toward the solution of the above mentioned conjecture and its extension to other non-Euclidean space forms. First, we prove (Theorem 1):

Any complete surface of non positive Gauss curvature isometrically immersed in \mathbb{R}^3 with one of its principal curvature functions k_i satisfying

$$k_i^2 \geq \text{const} > 0,$$

must be a generalized cylinder.

An interesting consequence of this result is, for instance, that generalized cylinders are the only complete surfaces with non positive Gauss curvature, isometrically immersed in \mathbb{R}^3 with mean curvature bounded away from zero (Corollary 1). As another

consequence, we generalize a result of Klotz and Osserman in [20] by showing that any complete special Weingarten surface in \mathbb{R}^3 on which the Gauss curvature does not change of sign is either a round sphere or a right circular cylinder. This gives a positive answer to a question raised by Sa Earp and Toubiana in [26].

Although an analogous to Efimov's Theorem in non-Euclidean space forms remains as an open problem to this day, a partial solution was obtained by Schlenker in [25]. Using a slightly different approach, the above mentioned results can be extended to other ambient spaces by application of the abstract theory of Codazzi pairs, that is, pairs (I, II) of real quadratic forms on an abstract surface, where I is a Riemann metric and II satisfies the Codazzi-Mainardi equations of the classical surface theory with respect to the metric I .

An abstract result about Codazzi pairs, see Theorem 6, let us to prove some consequences about immersions in the hyperbolic 3-space \mathbb{H}^3 of curvature -1 and in the sphere \mathbb{S}^3 of curvature 1 . Actually, we prove (see Corollaries 2 and 3):

No surface can be immersed in \mathbb{H}^3 (resp. \mathbb{S}^3) if it is complete in the induced Riemannian metric, with Gauss curvature $K \leq -1$ (resp. $K \leq \text{const} < 0$) and one of its principal curvature functions k_i satisfying

$$k_i^2 \geq \epsilon > 0,$$

for some positive constant ϵ .

About the geometry of surfaces with ends of non positive curvature in non Euclidean space forms we can prove, see Corollaries 4 and 5:

Consider a complete immersion in \mathbb{H}^3 (resp. \mathbb{S}^3) satisfying that outside a compact subset,

- *the Gauss curvature $K \leq -1$ (resp. $K \leq \text{const} < 0$) and*
- *$k_i^2 \geq \epsilon > 0$, $\epsilon \in \mathbb{R}$, where k_i is one of its principal curvature functions.*

Then it has finite total curvature and, in particular, it has finite topology and finite area.

2 A step in the solution of Milnor's conjecture

In this section we prove a partial solution to Milnor's conjecture and extend the result of Klotz and Osserman in [20] answering the question raised by Sa Earp and Toubiana in [26].

Along all the section we shall always assume that the differentiability used is \mathcal{C}^∞ but the differentiability requirements are actually much lower. Indeed, for most of the cases \mathcal{C}^3 -differentiability will be enough.

We also suppose that Σ is an oriented surface (otherwise we would work with its oriented two-sheeted covering).

First, we will show the following result holds:

Theorem 1. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a complete immersed surface of non positive curvature. If one of its principal curvatures k_i satisfies*

$$k_i^2 \geq \text{const} > 0,$$

then $\psi(\Sigma)$ is a generalized cylinder in \mathbb{R}^3 .

Proof. Let us denote by k_1 and k_2 the principal curvatures of ψ . Up to a change of orientation, we can assume that

$$k_2 \geq \epsilon > \frac{\epsilon}{2} > 0 > k_1, \quad \text{for some positive constant } \epsilon. \quad (2.1)$$

Consider ψ_ϵ the parallel map of ψ to a distance $2/\epsilon$, that is,

$$\psi_\epsilon := \psi + \frac{2}{\epsilon}N,$$

where $N : \Sigma \longrightarrow \mathbb{S}^2$ is the Gauss map of ψ . Then, it is not difficult to check that ψ_ϵ is an immersion which induced metric Λ_ϵ and element of area dA_ϵ are given by

$$\Lambda_\epsilon = I - \frac{4}{\epsilon}II + \frac{4}{\epsilon^2}III, \quad dA_\epsilon = -\frac{1}{\epsilon^2}(\epsilon - 2k_1)(\epsilon - 2k_2)dA,$$

where I , II and III denote the first, second and third fundamental forms of ψ and dA is the element of area of I . Moreover the principal curvature functions of ψ_ϵ , k_1^ϵ and k_2^ϵ can be written as

$$k_1^\epsilon = \frac{\epsilon k_1}{\epsilon - 2k_1},$$

$$k_2^\epsilon = \frac{\epsilon k_2}{\epsilon - 2k_2}.$$

Hence, the Gauss curvature, $K(\Lambda_\epsilon)$, of ψ_ϵ is given by

$$K(\Lambda_\epsilon) = \frac{\epsilon^2 K(I)}{(\epsilon - 2k_1)(\epsilon - 2k_2)} \geq 0, \quad (2.2)$$

where $K(I)$ denotes the Gauss curvature of I and writing Λ_ϵ in an orthonormal reference of principal vector fields $\{e_1, e_2\}$, we have that

$$\Lambda_\epsilon \equiv \frac{1}{\epsilon^2} \begin{pmatrix} (\epsilon - 2k_1)^2 & 0 \\ 0 & (\epsilon - 2k_2)^2 \end{pmatrix}. \quad (2.3)$$

From (2.1), (2.2) and (2.3), we deduce that $I \leq \Lambda_\epsilon$ and ψ_ϵ is a complete immersion in \mathbb{R}^3 of non negative curvature. Now, by using the Sacksteder theorem (see [24]), either $\psi_\epsilon(\Sigma)$ is a generalized cylinder or its Gauss curvature does not vanish identically and ψ_ϵ is a convex embedding satisfying one of the following items:

(A) Σ is homeomorphic to a sphere, or

- (B) Σ is homeomorphic to a plane and, up to a motion in \mathbb{R}^3 , there is a point $p_0 \in \Sigma$ such that the plane $\{z = 0\}$ is the tangent plane of $\psi_\epsilon(\Sigma)$ at $\psi_\epsilon(p_0) = (0, 0, 0)$ and the projection of $\psi_\epsilon(\Sigma)$ on $\{z = 0\}$ is a convex domain \mathcal{G} such that ψ_ϵ is a convex graph in the interior of \mathcal{G} , $\text{Int}(\mathcal{G})$, and a vertical segment at each point of $\mathcal{G} \setminus \text{Int}(\mathcal{G})$. Moreover, if $\{q_n\}$ is a divergent sequence of points in Σ , its height function $\{z(q_n)\}$ goes to infinity.

We will see that neither of these two cases can occur. In fact, the first case is not possible because any compact surface in \mathbb{R}^3 must have at least an elliptic point, which contradicts our assumption about ψ .

In the second case, it is clear that ψ_ϵ is a proper embedding. Actually, by the global convexity, there is a cone \mathcal{V}_ϵ with axis of rotation the OZ -axis such that $\psi_\epsilon(\Sigma)$ lies inside the cone \mathcal{V}_ϵ . But ψ is obtained from ψ_ϵ as a parallel surface to distance $2/\epsilon$, thus ψ is also a proper map and $\psi(\Sigma)$ lies inside the cone \mathcal{V} obtained as the parallel surface of \mathcal{V}_ϵ to a distance $2/\epsilon$.

Under these conditions we assert that ψ must have at least one elliptic point, which would also lead us to a contradiction. To see this, we can assume that, up to a vertical translation, the vertex of \mathcal{V} is the origin and as ψ is proper and it is contained in \mathcal{V} , the distance $\psi(\Sigma)$ to the origin is a positive real number $d_0 > 0$. Consider the spherical cap $\mathbb{S}_-^2(R, 2d_0)$ passing through the origin, of height $2d_0$ and boundary the circle of radius R obtained by the section $\mathcal{V} \cap \{z = 2d_0\}$. From the construction, $\psi(\Sigma) \cap \mathbb{S}_-^2(R, 2d_0) = \emptyset$. Thus, fixing this circle and taking the spherical caps $\mathbb{S}_-^2(R, 2d_0 - t)$ of height $2d_0 - t$, $0 \leq t \leq 2d_0$ passing through $(0, 0, t_0)$ with boundary the mentioned circle, we get a t_0 , $d_0 \leq t_0 < 2d_0$ where $\mathbb{S}_-^2(R, 2d_0 - t_0)$ intersects the surface $\psi(\Sigma)$ for the first time. It is clear that the intersection points must be interior and elliptic points of the surface. \square

As a consequence of the previous theorem, we have the following result:

Corollary 1. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a complete immersed surface with nonpositive Gauss curvature and mean curvature H verifying that $|H| \geq \text{const} > 0$. Then $\psi(\Sigma)$ is a generalized cylinder in \mathbb{R}^3 .*

Proof. It follows from the above theorem since the condition imposed on the mean curvature implies that one of the principal curvature functions k_i satisfies $k_i^2 \geq \epsilon > 0$ for some positive constant ϵ . \square

2.1 Complete special Weingarten surfaces

Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be an immersed surface with Gauss curvature K and mean curvature H . ψ is called a Weingarten surface if H and K are in a functional relationship $W(H, K) = 0$. We say that ψ is a *special Weingarten surface* (in short, SW-surface) if there exists a \mathcal{C}^1 -function $f : [0, \infty[\longrightarrow \mathbb{R}$ such that

$$H = f(H^2 - K), \quad f(0) \neq 0, \quad (2.4)$$

and

$$4tf'(t)^2 < 1, \quad \forall t \in [0, \infty[. \quad (2.5)$$

A function satisfying (2.5) is called an *elliptic function*.

It is remarkable that for any elliptic function f satisfying $f(0) \neq 0$, there is a round sphere of radius $1/|f(0)|$ in the family of SW -surfaces associated to f .

In [26] Sa Earp and Toubiana asked if the following result

KLOTZ AND OSSERMAN[20]. A complete surface in \mathbb{R}^3 with constant mean curvature on which the Gauss curvature K does not change sign is either a sphere, a minimal surface or a right circular cylinder,

can be extended to complete SW -surfaces. Here, we give an affirmative answer to this question and prove:

Theorem 2. *Let $\psi : \Sigma \rightarrow \mathbb{R}^3$ be a complete SW -surface on which the Gauss curvature K does not change sign. Then it is either a round sphere or a right circular cylinder.*

Proof. Assume the mean curvature H and the Gauss curvature K of the immersion ψ satisfy (2.4) for an elliptic function $f : [0, \infty[\rightarrow \mathbb{R}$. Then, if we denote by $t = H^2 - K$, the principal curvatures of ψ can be written as

$$k_1(t) = f(t) - \sqrt{t}, \quad k_2(t) = f(t) + \sqrt{t}, \quad (2.6)$$

and from (2.5), the following expression is satisfied

$$-\frac{1}{2\sqrt{t}} < f'(t) < \frac{1}{2\sqrt{t}}. \quad (2.7)$$

Moreover, up to a change of orientation if necessary, we can also assume that $f(0) > 0$.

CASE I: $K \leq 0$. In this case, as $f(0) > 0$ we find $t_0 > 0$ such that $f(t_0) = \sqrt{t_0}$ and by integration in (2.7), we obtain

$$-\sqrt{t} + 2\sqrt{t_0} < f(t) < -\sqrt{t}, \quad (2.8)$$

that is, $k_2 > 2\sqrt{t_0}$ on Σ (see Figure 2) and from Theorem 1, K vanishes identically. But, from (2.5), $h(t) := t - f(t^2)$ is a strictly increasing function and it has at most one zero. Thus, any complete SW -surface with vanishing Gauss curvature has constant mean curvature, that is, it must be a right circular cylinder, which concludes the proof in this case.

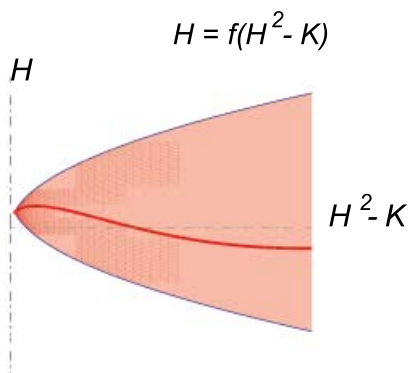


Figure 2: SW -surface with $K \leq 0$.

CASE II: $K \geq 0$. This case was proved in [2, Theorem 5]. □

Remark 1. CASE I in the above result has also been proved in [2] on the additional assumption that ψ is properly embedded.

3 An extension of Efimov's theorem

In this section we extend the results in [9, 10] to complete surfaces with ends of negative Gauss curvature. We will use the method developed by Efimov to prove that in \mathbb{R}^3 it is impossible to have complete \mathcal{C}^2 -immersed surfaces with Gauss curvature $K \leq c < 0$.

Throughout this section and as a standard notation, S will denote a surface with a compact boundary ∂S and Σ will denote a surface without boundary.

Definition 1. Assume that $\psi : S \rightarrow \mathbb{R}^3$ is a \mathcal{C}^2 -immersed surface with negative Gauss curvature $K < 0$. We say that the *reciprocal value of the curvature of ψ has variation with a linear estimate* if the following expression holds:

$$|\kappa(p) - \kappa(q)| \leq \epsilon_1 d_\psi(p, q) + \epsilon_2, \quad \forall p, q \in S, \quad (3.1)$$

for some non-negative constants ϵ_1 and ϵ_2 , where

$$\kappa = \frac{1}{\sqrt{-K}}, \quad (3.2)$$

and d_ψ denotes the distance associated to the induced metric.

Remark 2. The above class includes the family of surfaces with Gauss curvature bounded above by a negative constant. In fact, if $K \leq -\epsilon < 0$, then (3.1) holds for $\epsilon_1 = 0$ and $\epsilon_2 = 2/\sqrt{\epsilon}$.

The purpose of this section is to prove the following results:

Theorem 3. *Let S be a surface with a compact boundary ∂S and $\psi : S \rightarrow \mathbb{R}^3$ be a complete \mathcal{C}^2 -immersed surface with negative Gauss curvature. If the reciprocal value of the curvature of ψ has variation with a linear estimate, then*

$$\int_S |K| dA < \infty,$$

that is, ψ has finite total curvature and, in particular, S is parabolic and has finite topology.

Theorem 4. *Let $\psi : \Sigma \rightarrow \mathbb{R}^3$ be a complete \mathcal{C}^2 -immersion with negative Gauss curvature outside a compact set $C \subset \Sigma$. If the reciprocal value of the curvature of ψ has variation with a linear estimate outside C , then*

$$\int_\Sigma |K| dA < \infty,$$

that is, ψ has finite total curvature and, in particular, Σ is parabolic and has finite topology.

Theorem 5. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a complete \mathcal{C}^2 -immersed surface with Gauss curvature $K \leq -\epsilon < 0$ outside a compact subset C of Σ . Then Σ has finite topology, ψ is properly immersed and has finite area. Moreover, any end of ψ is asymptotic to a half-line in \mathbb{R}^3 (see Figures 1 and 3).*

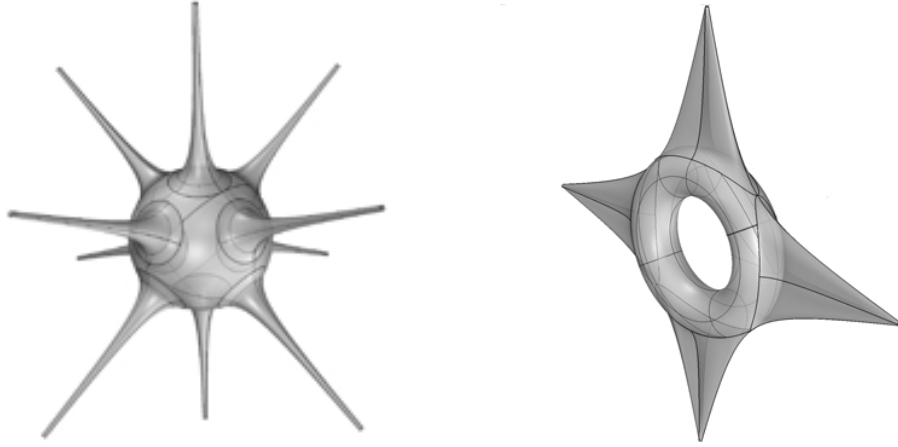


Figure 3: Complete surfaces with negatively curved ends.

3.1 The generalized lemmas

The proof of Efimov's theorem and its extension (see [9, 10]), are derived from a result which concerns special \mathcal{C}^1 -immersions $F : \mathcal{D} \longrightarrow \mathbb{R}^2$ where $\mathcal{D} \subset \mathbb{R}^2$ is constructed in [16, Section 2.1] as any open simply-connected region containing Ω ,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq \epsilon^2, \ y^2 \geq -cx\}, \quad \epsilon, c > 0,$$

and excluding the origin, (see Figure 4).

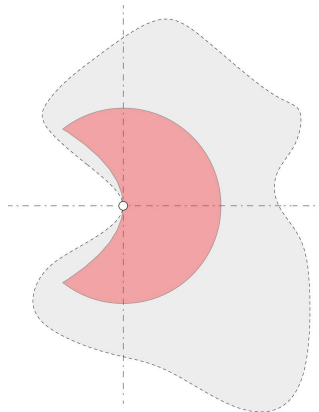


Figure 4: Open simply-connected region containing Ω .

On such a region \mathcal{D} we can consider the Riemannian metric g^* induced by F (that is, the g^* -length of an arc Γ in \mathcal{D} is just the Euclidean length of $F \circ \Gamma$) and its induced

distance d^* . On Ω , the following distance d_Ω^* can also be considered: $d_\Omega^*(p, q)$ is defined as the infimum of the g^* -lengths of rectifiable curves in Ω joining p to q for any $p, q \in \Omega$. It is clear that

$$d^*(p, q) \leq d_\Omega^*(p, q), \quad \forall p, q \in \Omega.$$

Although in [10] Efimov proved a more general version of the following Generalized lemma 1, we will formulate the result in a way that it can be used to prove the Generalized lemma 2.

Generalized lemma 1. *Let $F : \mathcal{D} \rightarrow \mathbb{R}^2$ be a potential given by $F = \nabla_0 f$, where $f : \mathcal{D} \rightarrow \mathbb{R}$ is a \mathcal{C}^2 -function satisfying*

$$\text{Det}(\nabla_0^2 f) \leq -\frac{1}{g^2} < 0, \quad (3.3)$$

$$|g(p) - g(q)| \leq \epsilon_1 d(p, q) + \epsilon_2, \quad \forall p, q \in \mathcal{D}, \quad (3.4)$$

for some positive function $g : \mathcal{D} \rightarrow \mathbb{R}^+$ and ∇_0 is the usual Euclidean gradient. Then the metric space (Ω, d_Ω^) cannot be complete.*

For a detailed discussion and proof of the above Generalized lemma 1 the reader is referred to [10, Generalized lemma A] and [16, Main Lemma].

Now, before formulating the Generalized lemma 2, we will clarify some notation and terminology following the same approach as in [16].

As we said at the beginning of this Section by S we shall denote a surface with a compact boundary ∂S and $\psi : S \rightarrow \mathbb{R}^3$ will be a complete \mathcal{C}^2 -immersed surface with negative Gauss curvature, $K < 0$. It is not a restriction to assume that S is orientable (in other case, we would work with its two-fold orientable covering).

Let $N : S \rightarrow \mathbb{S}^2$ be the Gauss map of ψ and consider III the third fundamental form of ψ , that is, the \mathcal{C}^0 -metric induced by N . By d_N we shall denote the distance associated to III .

Let (\tilde{S}, III) be the completion of (S, III) as a metric space and denote by $\partial \tilde{S} = \tilde{S} \setminus S$ the boundary set of \tilde{S} and by $\tilde{N} : \tilde{S} \rightarrow \mathbb{S}^2$ the continuous extension of N to \tilde{S} . As in [16], we can also introduce the following concepts:

Definition 2. Let Γ be a non geodesic open circular arc on \mathbb{S}^2 , $p \in \Gamma$ and $\epsilon > 0$. We consider from each point of Γ the open geodesic segment of \mathbb{S}^2 of length ϵ perpendicular to Γ and directed on the side of concavity of Γ . The region $R(\Gamma, \epsilon)$ formed by the union of Γ and all such segments is called an *exterior rectangle* of Γ at p . (See Figure 5).

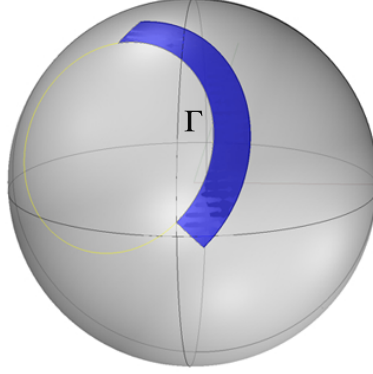


Figure 5: Exterior rectangle.

We call (\tilde{S}, III) *concave at a point* $p \in \partial\tilde{S}$ if p is in the closure \tilde{U} (in \tilde{S}) of an open region $U \subset S \setminus \partial S$ such that \tilde{N} is one-to-one on \tilde{U} and $N(U)$ contains the interior of an exterior rectangle at $\tilde{N}(p)$.

If (\tilde{S}, III) is not concave at any point of $\partial\tilde{S}$ we call (S, III) *pseudo convex*.

Generalized lemma 2. *Let $\psi : S \rightarrow \mathbb{R}^3$ be a complete \mathcal{C}^2 -immersion with negative Gauss curvature, $K < 0$. If the reciprocal value of the curvature of ψ has variation with a linear estimate, then (S, III) is pseudo convex.*

This result follows directly from the arguments of Efimov because the existence of a compact boundary, although it is not considered by him, in no way alters his proof. Indeed, it follows by taking into account the discussion about Lemma B (altered) in [10, Subsection 22] and observing that, even when the surface has a compact boundary, it is possible to apply the Generalized lemma 1 and complete the proof as in [9, Subsection 35].

3.2 Two auxiliary results

In this subsection we discuss two results used in the proof of Theorem 3. Although the same notations and terminology as in the previous section is used, we need to clarify some standard definitions and introduce further terminology.

Throughout, we shall consider on S the Riemannian structure induced by the third fundamental form III . We shall choose a compact subset C of S such that $\partial S \subset \text{Int}(C)$, by $\hat{\partial}C$ will denote the boundary set of $S \setminus \text{Int}(C)$ which is given by finitely many closed \mathcal{C}^1 -curves and d_N will be the distance associated to III .

Definition 3. If, locally, a parametrized arc on S is a shortest path between any two of its points, it is called a *geodesic arc*.

As N is a \mathcal{C}^1 -isometric immersion, we can check that any geodesic arc in S of length less than π is minimizing (i.e., it is the shortest path between any two of its points) and it is mapped by N one-to-one onto a path of equal length along a great circle on \mathbb{S}^2 .

If $p \in S \setminus \partial S$ and $\epsilon > 0$, we shall denote by $D_\epsilon(p)$ the geodesic disc of radius ϵ in S , that is,

$$D_\epsilon(p) = \{q \in S \setminus \partial S : d_N(p, q) < \epsilon\}.$$

$D_\epsilon(p)$ is called a *full* geodesic disc if one can leave p along a (half open) geodesic ray of length ϵ in every direction.

By a *convex* subset \mathcal{H} in S (or in \mathbb{S}^2) we understand a non empty subset which satisfies that any two of its points can be joined by a unique minimizing geodesic arc within \mathcal{H} (observe that with this definition \mathbb{S}^2 is not convex).

Lemma 1. *Consider a complete \mathcal{C}^2 -immersion $\psi : S \rightarrow \mathbb{R}^3$ with negative Gauss curvature, $K < 0$, r a positive real number such that $3r = d_N(\partial S, \hat{\partial}C)$ and $p, q \in S \setminus \text{Int}(C)$ two points satisfying*

$$d_N(p, q) < \min\{\max\{d_N(p, \hat{\partial}C), d_N(q, \hat{\partial}C)\} + r, \pi\}. \quad (3.5)$$

If the reciprocal value of the curvature of ψ has variation with a linear estimate, then there is a unique geodesic arc γ from p to q .

First, we remark the following assertion holds:

Assertion 3.1. Under the hypotheses of Lemma 1, if γ is a geodesic arc from p to q and $D_\epsilon(p)$ and $D_\epsilon(q)$ are two full geodesic discs in $S \setminus \partial S$ satisfying $l(\gamma) + 2\epsilon < \pi$ with $2\epsilon < r$, then there is an open convex subset \mathcal{H} in $S \setminus \partial S$ containing $D_\epsilon(p) \cup \gamma \cup D_\epsilon(q)$.

Proof of Assertion 3.1. Assume that $d_N(p, \hat{\partial}C) \geq d_N(q, \hat{\partial}C)$, then from (3.5), $D_\epsilon(p) \cup \gamma \cup D_\epsilon(q)$ lies on the geodesic disc $\mathbb{D} = D_{d_N(p, q) + 2\epsilon}(p)$ which is contained in $S \setminus \partial S$. Moreover, the convex subset \mathcal{H} can be constructed within \mathbb{D} using the same arguments as in [16, OBSERVATION 4, item (B)] and applying our generalized lemma 2 (instead of Lemma A of [16]). We detail here, for the reader's benefit, how this construction can be done.

CASE I. First, we make the construction of \mathcal{H} under the additional assumption that the closures of $D_\epsilon(p)$ and $D_\epsilon(q)$ in \tilde{S} lie within $S \setminus \partial S$.

Let $\mathcal{T}(\tau)$ be the closed tubular neighborhood of γ of radius τ inside \mathbb{D} . As N is a local diffeomorphism, it is clear there exists $\tau > 0$ such that N is one-to-one on

$$\overline{D}_\epsilon(p) \cup \mathcal{T}(\tau) \cup \overline{D}_\epsilon(q),$$

where by bar we denote the corresponding closure in S . Consider

$$\hat{\tau} = \sup\{\tau \in]0, \epsilon[: N \text{ is one-to-one on } \overline{D}_\epsilon(p) \cup \mathcal{T}(\tau) \cup \overline{D}_\epsilon(q)\},$$

then it is easy to see that $\hat{\tau} = \epsilon$, otherwise we find somewhere on the metric closure of $\mathcal{T}(\hat{\tau})$ a point $\tilde{p} \in \partial \tilde{S}$ such that $N(\tilde{p})$ lies on a non geodesics circle Γ on the boundary of $N(\mathcal{T}(\hat{\tau}))$ parallel to $N \circ \gamma$ with its center on the opposite side of Γ from $N(\mathcal{T}(\hat{\tau}))$. But then \tilde{S} is concave at \tilde{p} which gives a contradiction with the Generalized Lemma 2.

Now, we will consider that $N \circ \gamma$ parametrizes some portion of the equator $\{y = 0\}$ on \mathbb{S}^2 with its midpoint at $(0, -1, 0)$ and such that a certain $y_0 < 0$ is the y coordinate at the points $N(p)$ and $N(q)$. We will denote by \mathcal{R} the right elliptical cylinder in \mathbb{R}^3 formed

by the union of all lines parallel to the x -axis through the boundaries of $D_\epsilon(N(p))$ and $D_\epsilon(N(q))$. Let Π_τ^+ and Π_τ^- be the planes in \mathbb{R}^3 making an angle τ , $\tau \in [0, \pi/2]$, with $z = 0$ and tangent to \mathcal{R} along a line on which $y = \text{const} \geq y_0$ with $z = \text{const} \geq 0$ and $z = \text{const} \leq 0$, respectively, see Figure 6(a).

It is clear there exists a unique $\tau_0 \in]0, \pi/2[$ for which $\Pi_{\tau_0}^+$ and $\Pi_{\tau_0}^-$ pass through the origin in \mathbb{R}^3 and cut \mathbb{S}^2 along great circles tangent to the circular boundaries of $D_\epsilon(N(p))$ and $D_\epsilon(N(q))$.

On \mathbb{S}^2 we take the neighborhood \mathcal{E}_τ of $N \circ \gamma$ formed by the open region in $y < 0$ lying below Π_τ^+ , above Π_τ^- , to the right of $D_\epsilon(N(p))$ and to the left of $D_\epsilon(N(q))$, see the blue region in Figure 6(b).

From the first part of the construction, N is one-to-one on $D_\epsilon(p) \cup \mathcal{T}(\epsilon) \cup D_\epsilon(q)$ and N maps $D_\epsilon(p) \cup \mathcal{T}(\epsilon) \cup D_\epsilon(q)$ onto \mathcal{E}_0 . Let $\hat{\tau}$ be the supremum of all τ values in $[0, \tau_0]$ for which some neighborhood \mathcal{N}_τ of γ within \mathbb{D} is mapped by N one-to-one onto \mathcal{E}_τ , then we can prove that $\hat{\tau} = \tau_0$. Otherwise we find a point $\tilde{p} \in \partial \tilde{S}$ on the metric closure of $\mathcal{N}_{\hat{\tau}}$. But, under our assumption that the closures of $D_\epsilon(p)$ and $D_\epsilon(q)$ in \tilde{S} lie within $S \setminus \partial S$, $N(\tilde{p})$ is to distance greater than ϵ from $N(p)$ and $N(q)$. Thus, $N(\tilde{p})$ lies along the $\Gamma = \Pi_{\hat{\tau}}^+ \cap \mathbb{S}^2$ or $\Gamma = \Pi_{\hat{\tau}}^- \cap \mathbb{S}^2$ which centers lie on the opposite side of Γ from $\mathcal{E}_{\hat{\tau}}$ and \tilde{S} is concave at \tilde{p} which contradicts the Generalized lemma 2.

By taking $\mathcal{H} = \mathcal{N}_{\tau_0}$, it is clear that \mathcal{H} is a convex subset within \mathbb{D} and $N : \mathcal{H} \rightarrow \mathcal{E}_{\tau_0}$ is one-to-one.

CASE II. In the general case, we can apply for any $\tau \in]0, \epsilon[$ the Case I to $D_\tau(p) \cup \gamma \cup D_\tau(q)$ to get an open convex subset \mathcal{H}_τ within \mathbb{D} and containing $D_\tau(p) \cup \gamma \cup D_\tau(q)$. Then \mathcal{H} can be constructed by taking

$$\mathcal{H} = \bigcup_{0 < \tau < \epsilon} \mathcal{H}_\tau.$$

□

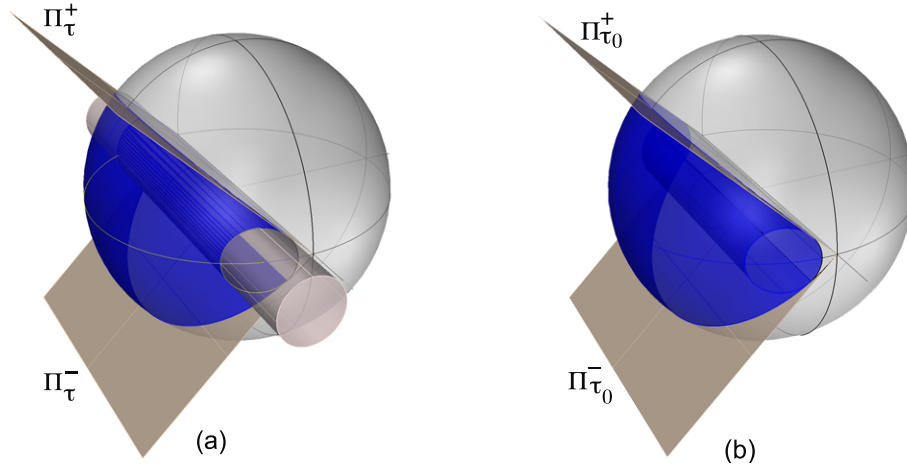


Figure 6: Construction of the convex subset $N(\mathcal{H})$.

Proof of Lemma 1. To prove the Lemma we suppose that $p \neq q$, otherwise it is trivial. Then, from (3.5) and since $S \setminus \partial S$ is connected, we can take a parametrized arc Γ

satisfying

$$l_N(\Gamma) < \min\{\max\{d_N(p, \hat{\partial}C), d_N(q, \hat{\partial}C)\} + r, \pi\},$$

where l_N denotes the length induced by the third fundamental form *III*.

But N is a local isometry and Γ is compact, thus there is a positive real number ϵ , $0 < \epsilon < r$ such that,

$$l_N(\Gamma) + 2\epsilon < \min\{\max\{d_N(p, \hat{\partial}C), d_N(q, \hat{\partial}C)\} + r, \pi\}, \quad (3.6)$$

and $D_\epsilon(x)$ is a full geodesic disc in $S \setminus \partial S$ for any point x of Γ .

Using again the compactity of Γ we can fix $x_0 = p, x_1, \dots, x_n = q$ points in Γ , ordered by the parametrization of Γ and satisfying

$$d_N(x_k, x_{k+1}) < \epsilon, \quad k = 0, \dots, n-1. \quad (3.7)$$

As $x_1 \in D_\epsilon(x_0)$ it is clear, there is a unique geodesic arc γ_0 within $S \setminus \partial S$ joining x_0 to x_1 with

$$l_N(\gamma_0) + 2\epsilon < \pi,$$

and, following the same ideas as in [16, *Proof of Lemma B*], we can apply an induction argument on the fixed number of points. More specifically, assume there is a unique geodesic arc γ_{k-1} within $S \setminus \partial S$ from x_0 to x_{k-1} with

$$l_N(\gamma_{k-1}) + 2\epsilon < \pi.$$

Then we prove the existence of a minimizing geodesic arc γ_k from p to x_k satisfying

$$l_N(\gamma_k) + 2\epsilon < \pi. \quad (3.8)$$

In fact, by applying the Assertion 3.1 to γ_{k-1} , there is an open convex set \mathcal{H}_{k-1} in $S \setminus \partial S$ containing $D_\epsilon(p) \cup \gamma_{k-1} \cup D_\epsilon(x_{k-1})$, but from (3.7), $x_k \in D_\epsilon(x_{k-1})$ and we find a minimizing geodesic arc γ_k from p to x_k within \mathcal{H}_{k-1} . Moreover, it is clear from (3.6) that (3.8) holds, which concludes the proof. \square

Lemma 2. $d_N(p, \hat{\partial}C) < \pi$ for any $p \in S \setminus C$.

Proof. We argue by contradiction. If the lemma does not hold, then from the compactity of C we can suppose there are $p \in S \setminus C$ and $q \in \hat{\partial}C$ such that $d_N(p, q) = d_N(p, \hat{\partial}C) = \pi$.

Take a positive real number $\epsilon < \min\{\pi/2, r\}$, such that $D_{2\epsilon}(q)$ is a full geodesic disc in $S \setminus \partial S$ and consider the circle $\mathbb{S}_\epsilon(q) = \partial D_\epsilon(q)$. Then, by fixing $q_1 \in \mathbb{S}_\epsilon(q)$ satisfying

$$d_N(p, q_1) = d_N(p, \mathbb{S}_\epsilon(q)) = \pi - \epsilon < \pi = \min\{\max\{d_N(p, \hat{\partial}C), d_N(q, \hat{\partial}C)\} + r, \pi\},$$

we can apply the Lemma 1 to q_1 and p and find a minimizing geodesic γ_1 in $S \setminus \partial S$ from q_1 to p . The geodesic ray from q to q_1 together γ_1 is a minimizing geodesic arc γ in $S \setminus \partial S$ joining q to p with $l_N(\gamma) = \pi$.

Let m be the midpoint of γ . For any t , $0 < t < \pi/2$, we denote by p_t and q_t the points in γ satisfying

$$d_N(p, p_t) = d_N(q, q_t) = t.$$

Throughout let us assume that $N(m) = (0, -1, 0)$, $N(p) = (-1, 0, 0)$, $N(q) = (1, 0, 0)$ and γ is mapped one-to-one into the corresponding geodesic arc of \mathbb{S}^2 in the plane $\{z = 0\}$.

We also choose $\epsilon_t > 0$, $\epsilon_t < \min\{t, r, \pi/2\}$ such that $D_{\epsilon_t}(p_t)$, $D_{\epsilon_t}(q_t)$ are full geodesic discs. Under these conditions we can apply the Assertion 3.1 and prove that there is an open convex subset \mathcal{H}_t in $S \setminus \partial S$ verifying that N maps \mathcal{H}_t one-to-one onto the convex subset $N(\mathcal{H}_t)$. But for the construction of \mathcal{H}_t (see the proof of Assertion 3.1 for more details) we can check that, see also Figure 6,

$$\lim_{t \rightarrow 0} N(\mathcal{H}_t) = D_{\pi/2}(N(m)) = \mathbb{S}^2 \cap \{y < 0\}, \quad (3.9)$$

and N maps $D_{\pi/2}(m)$ one-to-one onto $D_{\pi/2}(N(m))$ in \mathbb{S}^2 , see Figure 7.

At this point, we can take a positive real number $\epsilon_1 < \min\{\pi/2, r\}$, such that $D_{2\epsilon_1}(p)$ is a full geodesic disc and two different points $p_1, p_2 \in D_{2\epsilon_1}(p)$ to a distance ϵ_1 from p , which are mapped by N into the points $N(p_1)$ and $N(p_2)$ lying on the geodesic arc in the plane $\{y = 0\}$ in such a way that $N(p_1)$ lies in the half space $\{z > 0\}$ and $N(p_2)$ is lying in the half space $\{z < 0\}$, see Figure 7.

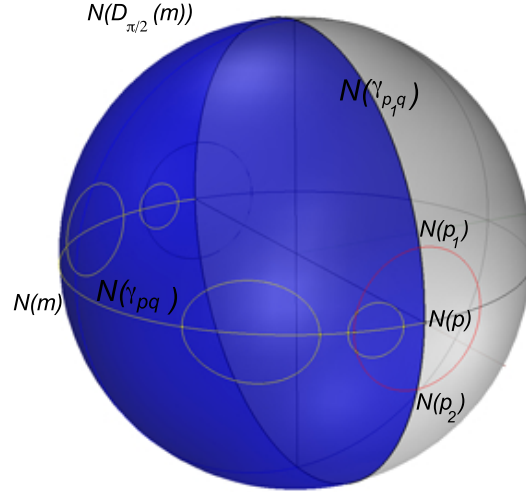


Figure 7: Proving the Lemma

By the choice of p_1 and p_2 we have $\pi - \epsilon_1 \leq d_N(p_i, q)$, $i = 1, 2$. Moreover, since N is a global isometry from $D_{\pi/2}(m)$ onto $\{y < 0\}$, there exist two sequences of curves $\{\Gamma_n^i\}$ joining q to p_i , $i = 1, 2$ which interior points lie within $D_{\pi/2}(m)$ and such that $l_N(\Gamma_n^i) \rightarrow \pi - \epsilon_1$. Thus,

$$d_N(p_i, q) = \pi - \epsilon_1 < \pi = \min\{\max\{d_N(p_i, \hat{\partial}C), d_N(q, \hat{\partial}C)\} + r, \pi\}, \quad i = 1, 2,$$

and we can apply Lemma 1 to prove the existence of minimizing geodesic arcs in $S \setminus \partial S$, γ_{p_1q} and γ_{p_2q} from p_1 to q and from p_2 to q , respectively. But, having in mind that there is a unique geodesic in \mathbb{S}^2 of length $\pi - \epsilon_1$ from p_i to q , $i = 1, 2$, we have that $N(\gamma_{p_1q})$ lies in $\{y = 0\}$ on the northern hemisphere of \mathbb{S}^2 and $N(\gamma_{p_2q})$ lies in $\{y = 0\}$ on the southern hemisphere of \mathbb{S}^2 .

Now, consider the following closed subset A_i in $\gamma_{p_i q}$,

$$A_i = \{x \in \gamma_{p_i q} \mid d_N(x, m) = \pi/2\}, \quad i = 1, 2$$

It is clear that there is a neighborhood of p_i in A_i . Moreover, by using that N is a local diffeomorphism, it follows that A_i is also an open subset of $\gamma_{p_i q}$ and so $A_i = \gamma_{p_i q}$, $i = 1, 2$. In other words, there is no point of $\partial \tilde{S}$ to a distance $\pi/2$ from m , the closure $\overline{D}_{\pi/2}(m)$ of $D_{\pi/2}(m)$ lies in $S \setminus \partial S$ and N is one-to-one in $\overline{D}_{\pi/2}(m)$. Since N maps $\overline{D}_{\pi/2}(m)$ one-to-one onto the eastern hemisphere of \mathbb{S}^2 while $\overline{D}_{\pi/2}(m)$ is compact there is $\epsilon_2 > 0$ such that $D_{\pi/2+\epsilon_2}(m)$ is a full geodesic disc and N maps it one-to-one onto $N(D_{\pi/2+\epsilon_2}(m))$. But $q \in \hat{\partial}C$ and $\hat{\partial}C$ is a finite set of regular curves in Σ , then by the above construction, we can assert that there are points of $N(\hat{\partial}C)$ in $N(D_{\pi/2+\epsilon_2}(m))$ which distance from p is less than π . This fact is a contradiction with the assumption that $\pi = d_N(p, \hat{\partial}C)$. \square

3.3 Proof of the Theorems 3, 4 and 5.

Proof of Theorem 3. To prove Theorem 3 we observe that from Lemma 2, $d_N(p, \hat{\partial}C) < \pi$, for any $p \in S \setminus C$, and from Lemma 1, there exists a minimizing geodesic arc, γ_p , joining p to $\hat{\partial}C$. By the minimizing property, γ_p meets orthogonally to $\hat{\partial}C$. Thus, the area of $\psi(S)$ respect to the third fundamental form III is given by

$$A_N(\psi(S)) = A_N(\psi(C)) + \int_0^{L_N(\hat{\partial}C)} l_N(\gamma_q) < A_N(\psi(C)) + L_N(\hat{\partial}C)\pi < \infty,$$

where $L_N(\hat{\partial}C)$ is the length of $\hat{\partial}C$ respect to III , which concludes the proof. \square

Proof of Theorem 4. It follows directly from Theorem 3. \square

Proof of Theorem 5. From Theorem 3, Remark 2 and the stated hypothesis in the theorem, the area of ψ , $A(\psi(\Sigma))$, is estimated by

$$A(\psi(\Sigma)) \leq \frac{1}{\epsilon} \int_{\Sigma} |K| dA < \infty.$$

That is, the immersion has finite area. Thus, we are left to show that every end of Σ is properly immersed and asymptotic to a half-line.

Let us consider an end of Σ , which we will assume parametrized on the set $E = \{p \in \mathbb{R}^2 \mid 0 < |p| \leq 1\}$. We can also assume that the curvature is non positive for every point on E . Since the area of the end is finite, there exists a strictly decreasing sequence of radii $\{\epsilon_n\}$ going to zero such that the curves $\Gamma_n = \psi(\{p \in \mathbb{R}^2 \mid |p| = \epsilon_n\})$ satisfy that their length

$$\{l(\Gamma_n)\} \rightarrow 0. \quad (3.10)$$

For $n < m$ denote by $\mathcal{A}_n^m = \psi(\{p \in \mathbb{R}^2 \mid \epsilon_m \leq |p| \leq \epsilon_n\})$ and $\mathcal{A}_n^\infty = \psi(\{p \in \mathbb{R}^2 \mid 0 < |p| \leq \epsilon_n\})$. Since the end has non positive curvature at every point, then

$$\mathcal{A}_n^m \subseteq \text{conv}(\Gamma_n \cup \Gamma_m) = \text{conv}(\text{conv}(\Gamma_n) \cup \text{conv}(\Gamma_m)) \quad (3.11)$$

(see, for instance, [23]), where $\text{conv}(\cdot)$ denotes the convex hull of a set in \mathbb{R}^3 .

Thus, as the end can not be bounded [7], $\cup_{n=1}^{\infty} \Gamma_n$ is unbounded. From this fact and (3.10), passing to a subsequence if necessary, we can assume

$$\max\{|p| \mid p \in \Gamma_n\} < \min\{|p| \mid p \in \Gamma_{n+1}\} \quad \text{with} \quad \{\min\{|p| \mid p \in \Gamma_n\}\} \rightarrow \infty. \quad (3.12)$$

Now, for all n we consider two points $q_n \in \text{conv}(\Gamma_1)$ and $p_n \in \text{conv}(\Gamma_{n+1})$. Then, passing to a subsequence if necessary, we can suppose there exists a unit vector v_0 in \mathbb{R}^3 such that

$$\left\{ \frac{p_n - q_n}{|p_n - q_n|} \right\} \rightarrow v_0.$$

Since $\{|p_n - q_n|\} \rightarrow \infty$, it is easy to check that the vector v_0 does not depend neither on the chosen points q_n because $\text{conv}(\Gamma_1)$ is a bounded set, nor on the chosen points p_n because the diameter of $\text{conv}(\Gamma_{n+1})$ goes to zero, from (3.10).

Let us define the solid cylinders

$$\mathcal{R}_1^+ = \{q + tv_0 \mid q \in \text{conv}(\Gamma_1), t \geq 0\}, \quad \mathcal{R}_1 = \{q + tv_0 \mid q \in \text{conv}(\Gamma_1), t \in \mathbb{R}\}.$$

And let us prove that $\mathcal{A}_1^\infty \subseteq \mathcal{R}_1^+$. From this condition, (3.11) and (3.12), we will have that the end is properly immersed.

Assume $\mathcal{A}_1^\infty \not\subseteq \mathcal{R}_1^+$ then, from (3.11), there exists $n_0 > 1$ and a point $x_0 \in \text{conv}(\Gamma_{n_0})$ such that $x_0 \notin \mathcal{R}_1^+$. Hence the compact set

$$\hat{C} = \left\{ \frac{x_0 - q}{|x_0 - q|} \in \mathbb{S}^2 \mid q \in \text{conv}(\Gamma_1) \right\}$$

does not contain to the vector v_0 .

Using (3.11), for each $n > n_0$ there exist two points $q_n \in \text{conv}(\Gamma_1)$ and $p_n \in \text{conv}(\Gamma_{n+1})$ such that $x_0 = (1 - t_n)q_n + t_n p_n$, for some $t_n \in [0, 1]$. In such a case $(p_n - q_n)/|p_n - q_n| \in \hat{C}$ which contradicts that the limit of this sequence must be v_0 .

Once we have proven that $\mathcal{A}_1^\infty \subseteq \mathcal{R}_1^+$ if, analogously, we define

$$\mathcal{R}_n^+ = \{q + tv_0 \mid q \in \text{conv}(\Gamma_n), t \geq 0\}, \quad \mathcal{R}_n = \{q + tv_0 \mid q \in \text{conv}(\Gamma_n), t \in \mathbb{R}\},$$

it is elementary to check that

$$\mathcal{A}_n^\infty \subseteq \mathcal{R}_n^+, \quad \text{for all } n > 1. \quad (3.13)$$

Since $\mathcal{R}_{n+1} \subseteq \mathcal{R}_n$, then from (3.10) we have that $\mathcal{R} = \cap_{n=1}^{\infty} \mathcal{R}_n$ is a line, and (3.13) proves that the end is asymptotic to \mathcal{R} as we want to show. \square

4 Complete surfaces with non positive extrinsic curvature in \mathbb{H}^3 and \mathbb{S}^3

Fundamental results of surfaces' theory in \mathbb{R}^3 essentially, only depend on the Codazzi equation which yields true if we consider any other space form. This is the case, for example, of Hopf's theorem on the classification of constant mean curvature spheres or Liebmann's theorem about surfaces of positive constant Gauss curvature. Thus, it is

not surprising that various results from theory of immersed surfaces can be proved in the abstract setting of Codazzi pairs, that is, pairs (I, II) of real quadratic forms on an abstract surface, where I is a Riemannian metric and II satisfies the Codazzi-Mainardi equations of the classical surface theory with respect to the metric I .

In this section, we use the theory of Codazzi pairs to give Efimov and Milnor's type results on surfaces in non euclidean space forms. The main idea in this sense is to study Codazzi pairs (I, II) as a geometric object in a non standard way on a Riemannian surface (Σ, I) of non positive Gauss curvature and use this study to deduce consequences when (I, II) are the first and second fundamental forms of a surface immersed in a space form. We shall follow the same approach as introduced by Aledo, Espinar and Gálvez, [2].

As in the previous sections, we shall assume that Σ is an oriented surface (otherwise we would work with its oriented two-sheeted covering). Moreover, throughout we always consider a \mathcal{C}^∞ -differentiability.

Definition 4. A fundamental pair on Σ is a pair of real quadratic forms (I, II) on Σ , where I is a Riemannian metric. The shape operator A of (I, II) is defined by

$$II(X, Y) = I(AX, Y), \quad X, Y \in T\Sigma. \quad (4.1)$$

We also define the *mean curvature*, the *extrinsic curvature* and the *principal curvatures* of the pair (I, II) as one half of the trace, the determinant and the eigenvalues of the endomorphism A . It is remarkable that, in general, there is not any connection between the extrinsic curvature of a fundamental pair (I, II) and the Gauss curvature $K(I)$ of the Riemannian metric I .

We say that the principal curvatures k_1 and k_2 of a fundamental pair on Σ are *strictly separated* if there exist real numbers c_1 and c_2 such that

$$k_1 \leq c_1 < c_2 \leq k_2, \quad (4.2)$$

on Σ .

Definition 5. Let (I, II) be a fundamental pair, we say that (I, II) is a *Codazzi pair* if the following equation holds

$$\nabla_X AY - \nabla_Y AX - A[X, Y] = 0, \quad (4.3)$$

for any vector field $X, Y \in T\Sigma$, where ∇ is the Levi-Civita connection of I .

Codazzi pairs appear in a natural way in the study of surfaces. For instance, the first and second fundamental forms of any surface isometrically immersed in a 3-dimensional space form is a Codazzi pair and the same happens for spacelike surfaces in a 3-dimensional Lorentzian space form.

In general, if a immersed surface in an n -dimensional (semi-Riemannian) space form has a parallel unit normal vector field ξ , then the first fundamental form and the second fundamental form associated with ξ constitute a Codazzi pair. Many other examples of Codazzi pairs also appear in [1, 2, 6, 18, 19, 21] and references therein.

Although we will apply our results on Codazzi pairs to surfaces in a 3-dimensional space form, all the above mentioned comments show that the results can also be applied to many others different contexts.

4.1 Codazzi pairs on complete surfaces with non positive curvature.

In this subsection we shall prove the following result:

Theorem 6. *Let (I, II) be a Codazzi pair on Σ with strictly separated principal curvatures. If (Σ, I) is a complete surface with Gauss curvature $K(I) \leq 0$, then only one of the following items hold:*

- *I is a flat metric and Σ is homeomorphic either a plane, or a cylinder or a flat torus.*
- *I is not flat, Σ is homeomorphic to a plane and*

$$\int_{\Sigma} |K(I)| dA_I \leq 2\pi. \quad (4.4)$$

Proof. Consider the third fundamental form III associated with the pair (I, II) , which is given by

$$III(X, Y) = I(AX, AY), \quad X, Y \in T\Sigma, \quad (4.5)$$

and take local doubly orthogonal coordinates (u, v) so that

$$\begin{aligned} I &= Edu^2 + Gdv^2, \\ II &= k_1 Edu^2 + k_2 Gdv^2, \\ III &= k_1^2 Edu^2 + k_2^2 Gdv^2. \end{aligned} \quad (4.6)$$

Such doubly orthogonal coordinates are locally available on an open dense subset of Σ and we can use them to check identities are valid on all the surface.

If $a \in \mathbb{R} \setminus \{0\}$ satisfies that $ak_1 \neq 1$, $ak_2 \neq 1$ on Σ , then from (4.6), the quadratic form

$$\Lambda_a = I - 2aII + a^2III$$

is a Riemannian metric given, locally, by

$$\Lambda_a = (1 - ak_1)^2 du^2 + (1 - ak_2)^2 dv^2, \quad (4.7)$$

which Gauss curvature, $K(\Lambda_a)$, can be written, see [17], as:

$$K(\Lambda_a) = \frac{K(I)}{(1 - ak_1)(1 - ak_2)}. \quad (4.8)$$

From (4.2), we can choose $a \in \mathbb{R} \setminus \{0\}$ so that

$$k_1 \leq c_1 < \frac{1}{a} < c_2 \leq k_2,$$

and, if we take $c_0 = \min\{|1 - ac_i| : i = 1, 2\}$, then from (4.7) and (4.8), the following expressions hold,

$$\begin{aligned} (1 - ak_i)^2 &\geq c_0^2, \quad i=1,2 \\ \Lambda_a &\geq c_0^2 I, \\ K(\Lambda_a) &\geq 0. \end{aligned}$$

That is, (Σ, Λ_a) is a complete Riemannian surface of non negative curvature.

We distinguish two cases:

CASE I: $K(\Lambda_a)$ vanishes identically. In this case, from (4.8), I is also a flat metric. Thus, if $\bar{\Sigma}$ denotes the universal cover of Σ , we have, from Cartan's theorem, that $(\bar{\Sigma}, I)$ is isometric to the usual euclidean plane \mathbb{R}^2 . But then, we deduce that Σ is homeomorphic to \mathbb{R}^2/Γ , where Γ is a discrete group of isometries acting properly on \mathbb{R}^2 , and the only possible oriented cases are the described ones in the first item of the theorem.

CASE II: $K(\Lambda_a)$ does not vanishes identically. In this case, we can consider on Σ the conformal Riemann structure induced by Λ_a and using Huber's results, see [15, Theorem 10, Theorem 12, Theorem 13], Σ must be conformally either a sphere or a plane. But if Σ is a sphere, from classical Gauss-Bonnet's theorem, (4.7) and (4.8), we have

$$4\pi = \int_{\Sigma} K(\Lambda_a) dA_{\Lambda_a} = - \int_{\Sigma} K(I) dA_I = -4\pi,$$

which gives a contradiction.

When Σ is not compact, it must be homeomorphic to a plane and from mentioned Huber's results, (4.7) and (4.8), we also have the following inequality:

$$\int_{\Sigma} |K(I)| dA_I = \int_{\Sigma} K(\Lambda_a) dA_{\Lambda_a} \leq 2\pi,$$

which concludes the proof. \square

From the proof of Theorem 6, we observe that if Σ has a compact boundary we can also apply Huber's results to get that (Σ, I) has finite total curvature. Actually, we can easily check the following result holds:

Theorem 7. *Let Σ be a surface and $C \subset \Sigma$ a compact subset. Assume (I, II) is a Codazzi pair on $\Sigma \setminus C$ which principal curvatures are strictly separated. If I is a complete metric with non positive Gauss curvature on $\Sigma \setminus C$, then $(\Sigma \setminus C, I)$ has finite total curvature. In particular, Σ is of parabolic type and has finite topology.*

4.2 Applications to non Euclidean space forms

In this subsection we apply the above Theorems 6 and 7 to obtain Efimov and Milnor's type results in the hyperbolic space, \mathbb{H}^3 , of sectional curvature -1 and in the sphere, \mathbb{S}^3 , of sectional curvature 1 .

Because Codazzi pairs' theory appear in the study of surfaces in other target spaces, analogous results could be given in many others contexts, for instance spacelike surfaces in the 3-dimensional Lorentzian space form or surfaces in an n -dimensional (semi-Riemannian) space form with a parallel unit normal vector field ξ .

As a first consequence of Theorem 6 we have,

Corollary 2. *Let $\psi : \Sigma \longrightarrow \mathbb{H}^3$ be an immersion with Gauss curvature, $K \leq -1$, and one of its principal curvature functions k_i satisfying,*

$$k_i^2 \geq \epsilon^2 > 0, \quad \text{for some constant } \epsilon > 0.$$

Then ψ is not a complete immersion.

Proof. Up to a change of orientation we can assume $k_2 \geq \epsilon > 0$. Then using the Gauss equation of the immersion,

$$k_1 k_2 = K + 1 \leq 0,$$

and we have that the principal curvatures of ψ satisfy $k_1 \leq 0 < \epsilon \leq k_2$. Thus, the pair (I, II) formed by the first and second fundamental forms of ψ is a Codazzi pair on Σ which principal curvatures are strictly separated. If we assume that I is complete, then by applying Theorem 6, we deduce that Σ is homeomorphic to a plane and its area is estimated as follows

$$A(\Sigma) \leq \int_{\Sigma} |K| dA \leq 2\pi,$$

which contradicts the well-known property that “any simply connected complete Riemannian surface of non positive Gauss curvature has infinite area”. \square

Corollary 3. *Let $\psi : \Sigma \longrightarrow \mathbb{S}^3$ be an immersion with Gauss curvature, $K \leq \text{const} < 0$, and one of its principal curvature functions k_i satisfying,*

$$k_i^2 \geq \epsilon^2 > 0, \quad \text{for some constant } \epsilon > 0.$$

Then ψ is not a complete immersion.

Proof. As above, we can assume $k_2 \geq \epsilon > 0$. Then, using the Gauss equation of the immersion,

$$k_1 k_2 = K - 1 < -1$$

and the principal curvatures of ψ satisfy the following relation $k_1 \leq 0 < \epsilon \leq k_2$. Now, the proof follows by applying Theorem 6 as in the above corollary. \square

Remark 3. A direct consequence of Theorem 6 is the non existence of complete immersed surfaces in \mathbb{H}^3 with Gauss curvature $K \leq \text{const} < 0$ and strictly separated principal curvatures.

Remark 4. By changing the hypothesis $K \leq \text{const} < 0$ by $K \leq 0$ and $\int_{\Sigma} |K| dA_I > 2\pi$ in Corollary 3 we have the same conclusion.

As straightforward consequences of Theorem 7, we also have,

Corollary 4. *Let $\psi : \Sigma \longrightarrow \mathbb{H}^3$ be a complete immersion and $C \subset \Sigma$ a compact subset. Assume that on $\Sigma \setminus C$ the Gauss curvature of ψ verifies $K \leq -1$ and one of its principal curvature functions k_i satisfies,*

$$k_i^2 \geq \epsilon^2 > 0, \quad \text{for some constant } \epsilon > 0.$$

Then ψ has finite area, Σ is parabolic and has finite topology.

Corollary 5. *Let $\psi : \Sigma \longrightarrow \mathbb{S}^3$ be a complete immersion and $C \subset \Sigma$ a compact subset. Assume that on $\Sigma \setminus C$ the Gauss curvature of ψ verifies $K \leq \text{const} < 0$ and one of its principal curvature functions k_i satisfies,*

$$k_i^2 \geq \epsilon^2 > 0, \quad \text{for some constant } \epsilon > 0.$$

Then ψ has finite area, Σ is parabolic and has finite topology.

Remark 5. One can apply the above corollaries to prove that in \mathbb{H}^3 (respectively, \mathbb{S}^3) any complete end with Gauss curvature $K \leq -1$ (respectively, $K \leq \text{const} < 0$) and one of its principal curvatures k_i satisfying

$$k_i^2 \geq \epsilon^2 > 0, \quad \text{for some constant } \epsilon > 0$$

has finite area and finite total curvature.

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